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Universal Dynamics of Complex Adaptive Systems: Gauge Theory of Things Alive *

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ABSTRACT

A universal dynamics of objects and their relations - a kind of *"universal chemistry"*- is discussed which satisfies general principles of locality and relativity. Einsteins theory of gravitation and the gauge theory of elementary particles are prototypes, but complex adaptive systems - anything that is alive in the widest sense - fall under the same paradigma. Frustration and gauge symmetry arise naturally in this context.
Besides a nondissipative deterministic dynamics, which is thought to operate at a fundamental level, a Thermo-Dynamics in sense of Prigogine is introduced by adding a diffusion process. It introduces irreversibility and entropy production. It equilibrates the chaotic local modes of the time development (only) and is designed to be undetectable under continued observation with given finite measuring accuracy. Compositeness and the development of structure can be described in this framework. The existence of a critical equilibrium state may be postulated which is invariant under the dynamics. But it is usually not reached in a finite time from a given starting configuration, because local dynamics suffers from critical slowing down, especially in the presence of frustration.

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1 Complex adaptive systems

are all those which are alive in the widest sense. They include chemical systems, especially autocatalytic ones, neural nets, living organisms, ecosystems, cognitive systems, organisations of human society such as universities, economies, etc.

All these systems consist of a multitude of individual agents - molecules, neurons, plants or animals, enterprises etc. Whatever they are, they organize into larger structures in a continuous process of adaptation and competition.

Striking similarities in the behavior of very different such systems have often been observed. Processes in the brain have been compared to the evolution of the species, for instance. Therefore a universal theory is sought. It must necessarily be quite abstract, since one wants to abstract from the concrete properties of special systems. Using such a universal theory as a framework, one hopes to generalize the results of model studies by reformulating them in such a way that only the notions available in the the general theory are used. This should then make it possible to translate to other situations, from physical chemistry to economy etc.

Chemistry is the science of atoms and the structure of their compounds. We seek a universal chemistry which deals with any kind of agents and their bonds.

The agents of a complex adaptive system evolve under the influence of the interaction with other agents. This is not so fundamentally different from what happens in physical systems. Therefore the general framework should accommodate the successful physical theories, in particular general relativity and the gauge theories of elementary particles. For simplicity only situations with a discrete number of agents will be considered here. What we find could be called a cross over between neural nets and lattice gauge theories. [1]

2 Paradigma

The fundamental principles are basically the same as in the prototypical physical theories. The theory has therefore a geometric character. [2]

1. **Locality** or *Nahewirkungsprinzip*: The relations imposed by fundamental laws are local.

   The fundamental physical equations are valid everywhere but they relate only dynamical variables at the same point of space and time, or in an infinitesimal neighbourhood of it.

2. **Relativity** or *Naheinformationsprinzip*: There exists in general no a priori possibility to compare physical quantities of observers (or "agents") at different positions in space time.

   To compare, a signal must be emitted. It will in general be influenced by the medium or by intermediary stations through which it passes.

   An agent manifests itself only through its (direct or indirect) relations with other agents.

3. **emergence** Out of the presence of local relations there arise collective phenomena of a nonlocal character.

[1] General relativity can be considered as a gauge theory, [42, 18] and there exist discretized versions of it [44, 43]

[2] This reminds of Spinoza. His opus *Ethica, ordine geometrica demonstrata* influenced Lessing and Goethe.

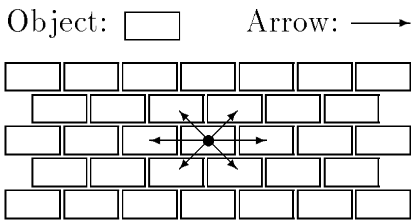


Figure 1: brick wall

The word "emergence" is not costumary in physics, but it makes sense. Examples are the propagation of electromagnetic waves or the gravitational force exerted by the sun on the earth.

Special methods have been developed for cases, where an immediate solution of the fundamental equations is not possible. Among them is Wilson's renormalization group. It furnishes a general method by which to derive effective theories which describe phenomena on coarser and coarser length scales. [21]. In addition, multigrid and similar multiscale methods exist also for the solution of partial differential equations [23] and for other problems including quantum field theory [24].

The importance of a relativity principle may not be obvious in the context of complex adaptive system. To make the point, let me formulate the

postmodern relativity principle:*Reality is of no immediate importance for human action. Only its perception is important, i.e. what is held to be reality*

True or not, it is certain that this principle is operative in politics and elsewhere. This shows that relativity principles are not irrelevant to our problem.

3 Structure: What is a thing? Why is it more than the sum of its parts

Let us first consider structure in inanimate things. Consider a brick wall as an example (figure 1). It consists of objects, the bricks, which are its parts. The arrows in figure 1 indicate neighbourship relations. These arrows determine the structure. They make the wall into more than the sum of its bricks.

In mathematics, objects and arrows make a category. We consider things as categories in this sense. The arrows determine their structure. The objects of a category can be categories themselves. That is, they may have internal structure themselves. In this way things can be made out of things. A tower may be built out of brick walls, for instance.

Without loss of generality, all objects may be regarded as categories, since every object may be made into a trivial category. It has only one object and one arrow which represents the identity of the object with itself.

In figure 1, only the arrows to nearest neighbour bricks were drawn. They may be thought to represent the spatial translation which takes one brick into the other. We may also consider arrows to next nearest neighbours etc. They are obtained by composition of arrows to nearest

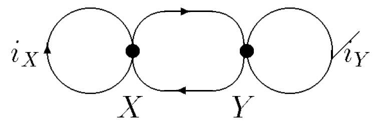


Figure 2: “couple”

neighbours, etc.

Categories

The axiomatic properties of a category are as follows:

A category K consists of a number of *objects* $X, Y, ...$, and of *arrows* $f, g, ...$ which point from one object to another. [3] We write $Obj(K)$ for the class of objects of a category K, and $Mor(Y, X)$ or $Mor_K(Y, X)$ for the set of its arrows from one object X to another object Y. In place of $f \in Mor(Y, X)$ the alternative notation $f : X \mapsto Y$ is used. There are arrows from one object X to itself. Among them is ι_X which expresses the relation of identity of the object with itself

1. (composition of arrows:) If $f : X \mapsto Y$ und $g : Y \mapsto Z$ are arrows then the arrow
$$g \underset{Y}{\circ} f : X \mapsto Z$$
is defined. The composition is associative.

2. *(identity)* ι_X is defined for all objects X, and $\iota_Y \underset{Y}{\circ} f = f \underset{X}{\circ} \iota_X$ for all $f : X \mapsto Y$.

The symbol Y underneath $\circ$ could be omitted, because an arrow f in a given category can point to one object only. But the chosen notation will be convenient for our purpose, because we shall have to consider several categories at the same time. Moreover we insist on writing ι_X rather than id for the arrow which expresses identity.

An example of a category with two objects is shown in figure 2. It represents a couple. X is married to Y and Y to X, and each of them is identical with itself.

Examples from mathematics

1. objects X: topological spaces
   arrows $f : X \mapsto Y$: continuous maps

2. objects X: commutative groups
   arrows $f : X \mapsto Y$: homomorphisms of groups

In commutative groups, the group multiplication is usually written as $+$. Examples of commutative groups are $\mathbf{Z}$ (integers), $\mathbf{Z}_p$ (integers modulo p), $\mathbf{Z} \times ...\mathbf{Z} \times ...\mathbf{Z}_{p_1}...\mathbf{Z}_{p_n}$.

[3] In German, the terminology *Pfeil*=arrow is standard. In english, the name *morphism* is more often used. We stick to “arrow”

Application to complex adaptive systems

We formulate a calculus of agents and their relations in the language of categories.

$$\text{objects} = \text{agents} \tag{1}$$
$$\text{arrows} = \text{relations, connections or bonds between agents} \tag{2}$$

This provides an extremely general framework. Yet it will be seen to lead straight to gauge theories - more precisely to a kind of crossing of neural nets with lattice gauge theory.

The arrows are directed. A mutual relation will have to be represented by a double arrow, one arrow in one direction and one into the other.

The composition property of arrows is reasonable in the context of complex adaptive systems. Think of friends of a friend.

We shall impose additional structure later on. Certain arrows will be declared elementary. All the others may be composed from them. In the brick wall, the arrows to nearest neighbours would be the elementary ones. The identity arrow is also considered elementary.

4 Dynamics

Dynamics is time development. Agents (objects) will change under the influence of their relations. And the relations (arrows $f : X \mapsto Y$) will change by the action of the agents concerned (X and Y). We regard those relations f as elementary which can effect a change of agent Y without time delay. This will be made more precise in a moment.

The time development of agents and their relations determines a map

$$t \mapsto K_t$$

to a category which depends on time. Following a suggestion of Sorin Solomon's I call this a *drama*.

The agents will typically have some degree of permanence and a history. Therefore it is reasonable to speak of a change of the *state* of an object with label or address x, or "at" x for short. Sometimes x is called a *node*. Similarly elementary arrows will be labelled by labels b called links. $b = (y, x, \beta)$ if the associated arrow points from an object at x to an object at y.

At a level of fundamental objects we may imagine a dynamics which is deterministic and nondissipative. Let us assume that it is Hamiltonian, with continuous time t.

We will discuss later on how a stochastic and dissipative dynamics emerges from this at the level of composite objects, i.e. objects with internal structure.

Let $\mathcal{M}$ be the space of all possible states of the collection of objects and arrows, i.e. $K_t \in \mathcal{M}$ for all times t. Call it phase space. Suppose that coordinates $\{\xi^\alpha\}$ are chosen in some way to identify points $\xi \in \mathcal{M}$. According to the relativity principle, there are no preferred coordinates on $\mathcal{M}$, and the dynamical laws must be formulated so that they are valid for any choice of coordinates. We demand, however, that each coordinate ξ^α should refer to only one object or arrow.

The Hamiltonian equations have the form ($\partial_\alpha = \partial/\partial\xi^\alpha$)

$$\frac{d}{dt}\xi^\alpha = \sum_\beta \partial_\beta H(\xi)\omega^{\beta\alpha} \tag{3}$$
$$= \{H, \xi^\alpha\} \tag{4}$$

with Hamiltonian H, symplectic matrix ω and Poisson brackets $\{,\}$.

$$\begin{aligned} \omega^{\alpha\beta} &= -\omega^{\beta\alpha} \\ \{F,G\} &= (\partial_\alpha F)\omega^{\alpha\beta}(\partial_\beta G), \end{aligned} \tag{5}$$

Example: In a mechanical system with a 2-dimensional phase space, $\xi^1 = p, \xi^2 = q, \omega^{12} = 1 = -\omega^{21}$, and (3) are the usual Hamilton equations of motion.

We do *not* insist that the matrix ω has an inverse. If it has not, we speak of a *degeneracy.* There are very interesting models which are degenerate in this sense. Degeneracies lead to constraints, as we shall see.

Maxwell's electrodynamics on a lattice, formulated in terms of electric and magnetic fields, are an example, and there are many similar ones. The symplectic matrix is given by the incidence matrix of a simplicial or cell complex in these examples, and the degeneracy comes from the fact that the boundary operator has square $\partial^2 = 0$ (see appendix B). Lattice gauge theories in a first order formalism are also degenerate in this sense. Applications of lattice gases to model predator prey systems have been proposed [46]

$\omega^{\alpha\beta}$ may be ξ-dependent. In this case it must satisfy conditions which ensure that the Poisson brackets satisfy the Jacobi identity, and that a volume form v on $\mathcal{M}$ exists which is invariant under the Hamiltonian time development

$$\{\{F,G\},K\} + \{\{K,F\},G\} + \{\{G,K\},F\} = 0 \,, \tag{6}$$

$$\int dv\, \{F,H\} = 0 \tag{7}$$

for every F. If $(\omega^{\alpha\beta})$ has an inverse $(\omega_{\alpha\beta})$ then the condition is sufficient that the 2-form $\Omega = \omega_{\alpha\beta} d\xi^\alpha d\xi^\beta$ is closed [32].

The Poisson bracket defines a derivation of the algebra of functions on $\mathcal{M}$,

$$\{F, GK\} = \{F,G\}K + G\{F,K\} \;. \tag{8}$$

If a deterministic dynamics is given, then the initial state ξ_0 of the system at some time $t = 0$ determines the state ξ_t for all later times t.

NB: The initial state is considered known. It is the aim of the formulation of a dynamical law to determine the future state from a *known* initial state. How to know it is another question.

In a *universal dynamics* one seeks to extract as much information as possible from the known initial state. The Hamiltonian (or the principles from which it is determined) should be as universal as possible, while different model situations are distinguished by initial states of different kinds.

For instance, the Maxwell equation

$$\nabla E = 4\pi\rho \tag{9}$$

is a condition on the possible initial states. The charge density ρ is proportional to the charges of the particles which serve as coupling constants in the Hamiltonian. Therefore the initial state contains information on them.

It is characteristic of gauge theories [12] that the initial state must satisfy constraints which fix conserved local quantities.

$$\Phi_i = 0 \tag{10}$$

$$\{H, \Phi_i\} = 0 \;. \tag{11}$$

It is standard to call also Φ_i a constraint. We distinguish

1. Degeneracy constraints: They satisfy

$$\{\Phi, \xi^\alpha\} = 0$$

for all variables ξ^α

2. Generators of infinitesimal gauge transformations $\xi \mapsto \xi + \delta\xi$,

$$\delta\xi^\alpha = \{\Phi, \xi^\alpha\} \tag{12}$$

It follows from the definition (12) that infinitesimal gauge transformations leave the Hamiltonian invariant,

$$\frac{d}{d\epsilon} H(\xi + \epsilon\delta\xi)|_{\epsilon=0} = -\{H, \Phi\} = 0 \ . \tag{13}$$

Of course, the values of all conserved quantities are determined by the initial conditions. If the Hamiltonian is not explicitly time dependent, then energy is conserved. Therefore the motion is constraint to the energy surface $\mathcal{M}_E$ where

$$H = E \ .$$

In general relativistic theories, $E = 0$ is often enforced by other constraints.

Locality

We exhibit additional structure in categories which describe complex systems. Some of the arrows will be declared *elementary*. They represent direct relations. Other arrows which represent indirect relations can be composed from them.

This substitutes for locality properties in physics. Fundamental physical laws relate only physical quantities in an infinitesimal neighbourhood of the same point in space and time. In discretized models such as lattice gauge theory, this is embodied through nearest neighbour interactions.

In complex adaptive systems the situation can be more complicated. The local topology can be complicated. And at a high level, where agents have much internal structure, there can be special agents which are connected to all or a large fraction of the other agents. This is called broadcasting. Media (radio, TV, newspapers) are examples.

But similar situations arise also in physics. Newtons law of gravitation involves forces at a distance. Yet Newtons law can be derived (modulo very small corrections) from Einsteins general relativity which obeys very strong principles of locality. Spacetime becomes a medium here.

It has been noted that seemingly purposeful activity of living organisms need not be based on an organizing intelligence. The fundamental idea is that the agents (e.g. legs of an insect) influence each other only locally, from neighbour to neighbour, and this results in global behaviour which is not governed by a central intelligence. Studies of insects showed for some animals that their legs exchange neuronal signals, and this results in coordinated walking, without need for an insect brain which coordinates the activity [48]

Ultimately we want to understand all kinds of organized behavior and self organization as a consequence of local interactions at some fundamental enough level. This is what the word *emergence* means. The examples show that often seemingly high level behavior can readily be understood as a consequence of local interactions.

This principle is being discussed in many disciplines. Another example is the detection of saliency in pictures by the visual system. Ullmann and Shashua presented a model which explains how this task could be performed locally in low level neurons in the visual cortex [13]. The salient features of a picture are characterized by relatively long straight lines. Thinking of the picture as made of directed line elements (which are recorded by Hubel-Wiesel neurons [14]), the activity of a neuron which represents such a a line element gets enhanced in each step of an iterative process if it has neighbours of nearly the same direction. The iteration steps can be regarded as local dynamics in discrete time.

Let us return to the formalism of categories. The formulation of a dynamics forces us to consider families of categories made out of objects and arrows which are taken from certain sets (state spaces). To prepare the ground for a local dynamics we need to specify the locality properties of the composition law of arrows first.

Postulate 1 (Locality of the composition law) *$f \underset{X}{\circ} g$ is defined if f, g, X are elements of at least one category K in the family. It depends on f, g, X but not on K.*

The above mentioned condition of locality on the dynamics will be satisfied if the Hamiltonian and the symplectic form have the following properties.

The state of the object at x will be determined by real variables ξ^{xm}, and the state of the arrow at link b by real variables ξ^{bn} Collectively they are denoted $\xi = (\xi^{\alpha})$, $\alpha = (x, m)$ resp. (b, n).

Postulate 2 (First order formalism)

$$H(\xi) = \sum_b H_b(\xi), \quad . \tag{14}$$

and the following restrictions are imposed on the ξ-dependence of H_b and on the symplectic matrix $\omega^{\alpha\beta}$.

1. *H_b depends only on the state of the arrow at link b and on the state of the objects which it links;*

2. *$\omega^{\alpha\beta}$ depends only on the state of those objects and arrows to which α, β refer; in case of an arrow it may also depend on the object to which this arrow points. $\omega^{\alpha\beta} = 0$ unless α, β refer to the same object or arrow, or to an arrow and the object to which it points.*

ι_X may occur among the arrows in the sum (14). In this case we may write H_x in place of H_b. H_x depends only on the state of object X at x.

Apart from locality, the form of H embodies the assumption that the "force" on an object is the sum of forces which are exerted by objects with which it has direct relations. That is, there are no true "multi-object-forces". This postulate refers only to the fundamental deterministic dynamics. The effective stochastic and dissipative dynamics at the level of composite objects with internal structure will not inherit this property.

Let O be some subset of all objects, and let $K(O)$ be the category which is generated by these objects and the elementary relations between them. Starting from the Hamiltonian form of the equations of motion, and from the locality properties of the Hamiltonian, the time development of $K(O)$ under the influence of its environment can be analysed. This is done in Appendix D. The environment consists of objects not in O, together with their elementary relations.

Kinematics

Besides the dynamical laws, there can also be kinematical rules. They will determine changes of status of arrows from composite to elementary and vice versa. In physical systems this may be determined by neighbourhood of objects in space.

5 Communication Networks, Consensus, Gauge invariance

Let us return to mathematics for a moment. Given categories K and K', a *functor* F is a map

$$K \mapsto K'$$

which maps objects into objects and arrows into arrows in such a way that the composition rules for arrows are respected

$$\begin{aligned} F(g \underset{X}{\circ} f) &= F(g) \underset{F(X)}{\circ} F(f) , & (15) \\ F(\iota_X) &= \iota_{F(X)} . & (16) \end{aligned}$$

There are also contravariant functors where the order is reversed, but we will not need them.

Example from mathematics

Algebraic topology exhibits a functor from the category of topological spaces and continuous maps to the category of commutative groups and homomorphisms [50]

This shows that functors can exhibit common structure of completely different categories. This is exactly what we want.

Communication networks

We say a category K' is a representation of K if there exists a functor $K \mapsto K'$.

Theorem 3 (Representation of a category as a communication network) *Every category K permits a faithful representation with the following properties*

To every object X there exists an input space A_X and an output space Ω_X. The input space contains a distinguished element $\emptyset$ ("empty input"). Arrows $f \in Mor(Y, X), g \in Mor(Z, Y)$ and objects X act as maps

$$\begin{aligned} X : A_X &\mapsto \Omega_X, & (17) \\ \iota_X : \Omega_X &\mapsto A_X & (18) \\ f : \Omega_X &\mapsto A_Y & (19) \end{aligned}$$

with the properties

$$X\iota_X = id : \Omega_X \mapsto \Omega_X \quad , \quad \iota_X X = id : A_X \mapsto A_X \ , \qquad (20)$$

$$g \underset{Y}{\circ} f = gYf : \Omega_X \mapsto A_Z \ . \qquad (21)$$

It should be noted that ι_X does not act as the identity map in general. In the context of cognition, ι_X may be associated with *consciousness*. It makes an agent aware of his actions (output).

The proof of the theorem will be given in Appendix A.

Let us note that this theorem refers to a single category, and the additional structure brought in by locality is not considered. In fact, the construction of the communication network uses global properties of the category.

When the locality postulates are imposed, a composite arrow f which is incident on Y may have an implicit dependence on Y. This occurs when the breakdown of f into elementary arrows contains a segment $f_n \underset{Y}{\circ} f_{n+1}$. One may extract this Y dependence and use it to define a composition law for inputs, under certain extra conditions.

Curvature = Field Strength = Frustration

$$\begin{tikzcd} & Y_1 \arrow[dr, "g_1"] & \\ X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & Z \\ & Y_2 \arrow[ur, "g_2"'] & \end{tikzcd}$$

This concept is central in gauge theories of all kinds. It is called curvature in general relativity, field strength in gauge theories of elementary particles, and frustration in spin glasses. But it is all the same and can be formulated in the general context of categories.

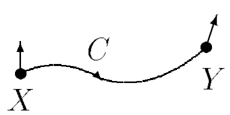
Relativity

In general relativity there is a space time manifold $\mathcal{M}$ with points x. Particles travel along worldlines on $\mathcal{M}$. These world lines may be parametrized by eigentime τ. The eigentime is shown by an ideal clock which the particle carries along. The worldline penetrates each spacelike hypersurface Σ ("space") only once.

The 4-velocity $u \in T_x\mathcal{M}$ of the particle at x is defined as the tangent vectors to its worldline, parametrized by eigentime. $T_x\mathcal{M}$ is the linear space spanned by all tangent vectors to curves through x.

There exists a basic principle of locality and relativity which generalizes the famous Nahewirkungsprinzip:

Naheinformationsprinzip: [18]
There is no a priori possibility of comparing vectors $u \in T_x\mathcal{M}$ and $v \in T_y\mathcal{M}$ at distinct points $x \neq y \in \mathcal{M}$. To compare, u must be parallel transported along a path C from x to y, for instance by emission of a signal which travels along C, i.e. by a process of communication



The result of the parallel transport will in general depend on the path C and on the gravitational fields encountered at intermediate stations z on C.

The path C may be considered composed from infinitesimal pieces $C = b_n \circ ... \circ b_1$, and the parallel transporter

$$U(C) : T_x\mathcal{M} \mapsto T_y\mathcal{M} \tag{22}$$

may be regarded as a composite arrow

$$U(C) = U(b_n) \circ ... \circ U(b_1) \tag{23}$$

which is made from elementary arrows attached to links b_i. After a choice of coordinates on $\mathcal{M}$ and a of a corresponding holonomic basis in $T_x\mathcal{M}$,

$$U(b) = 1 - \Gamma_\mu(x)\delta x^\mu, \ . \tag{24}$$

with matrix $\Gamma_\mu(x)$ furnished by the connection coefficients (Christoffel symbols).

Curvature (or frustration) is present if the parallel transporters have a nontrivial path-dependence for given initial and final points of the path C.

In general relativity, there is a continuum of points x. But there exist discretized versions of it with a countable number of points x and of links b from which paths can be composed [43, 44].

Generalization to the general situation is possible because arrows in a category may be composed. The Naheinformationsprinzip has its counterpart in the theory of complex adaptive systems:

Postulate 4 (Relativity principle) *There is no a priori way of comparing the state of two different agents. Any one of them manifests itself only through its communication with the others*

In our framework, this communication is described by arrows which may be composed.

The relativity aspect has been stressed by J. Holland [31] His *Darwin relativity principle* asserts the following. Each agent attempts all the time to adapt to all the others.

There is no absolute measure of fitness which would be maximized by an evolutionary process, but only a fitness relative to an environment. One speaks therefore of *coevolution.*

In computer simulations, adherence to strict standards of data encapsulation in object oriented programming [47] offers some protection against inadvertent violation of the relativity principle.

Frustration can occur in categories. In the following pictorial example it occurs, if the two composite arrows from X to Z are different.

$$\begin{tikzcd} & Y_1 \arrow[dr, "g_1"] & \\ X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & Z \\ & Y_2 \arrow[ur, "g_2"'] & \end{tikzcd}$$

Definition 5 (Unfrustrated category) *A category is unfrustrated if there never exist two different arrows from one given object X to another one, Y. In particular there is only one arrow ι_X from X to X.*

When elementary arrows are singled out in the given category K, a path C from x to y is specified by a sequence of links $b_1...b_n$ again, with $b_k = (x_k, x_{k-1}, \alpha_k)$, $x_n = y$, $x_0 = x$. The elementary arrows attached to them may be composed. In general, the resulting arrow may depend on the path C. The category is unfrustrated if it does not.

If a category is unfrustrated, synchronization is possible which produces consensus about when two agents are in the same state.

With frustration, one agents owl is the other's nightingale. Generally speaking, frustration is the simultaneous presence of relations with contradictory tendencies. This has consequences similar to what is familiar in spin glasses. Here is a description of a "model economy" to illustrate the point.

Example 6 (frustrated economy) *The example is based on the model assumption that it is man's fundamental need to feel superior to his neighbours. If so, the absolute level of material welfare is not essential. What counts in an economy is that every consumer possesses more than his neighbours do. Economy appears then as an iterative procedure which aims at satisfying this condition as well as possible.*

This is a frustrated system in the sense that the dynamics is governed by relations which are contradictory in their tendencies. As a result, there is no well-defined global optimum which the iteration would approach. (Equality would satisfy nobodies needs...). If one were to express by a cost functional the degree to which all the agents' needs are satisfied, its minima would be very sensitive to changes in neighbourhood relations and to the precise form of the cost functional (e.g. by a change of perception by the agents of their situation as could occur under the influence of the media).

Taking account of the possible back reaction of the economic development on these parameters, an autopoietic picture [10, 11] of fluctuations which generate fluctuations becomes plausible.

Traditional economic theory is badly equipped to deal with situations as in the example. It has been criticized for similar reasons by Arthur [28].

The curvature- or field strength tensor

describes the path dependence locally. Consider a closed path C from x to x made of four links $b_1, ...b_4$. In an unfrustrated category, the arrow f attached to C should be ι_X, where X is the object at x. Therefore

$$Xf = id : \Omega_X \mapsto \Omega_X$$

in the language of communication networks. The frustration is measured locally by how much this condition is violated.

In general relativity and standard gauge theories, the parallel transporters $U(C) = Xf$ are linear maps. In this case one may consider the difference $Xf - id$. This defines the field strength tensor.

Suppose that coordinates $\{x^\mu\}$ have been introduced on the space of points x such that the links b_1, b_2 are paths along coordinate lines in μ, ν direction respectively, and similarly for b_3, b_4 (see figure 3). Then one defines the components $F_{\mu\nu}$ of the field strength tensor by

$$U(C) = 1 - F_{\mu\nu}(x)\delta x^\mu \delta x^\nu \quad \text{(no sum)} \tag{25}$$

In the general nonlinear case one has only the composite arrows along closed curves as a substitute for $1 - F_{\mu\nu}\delta x^\mu \delta x^\nu$.

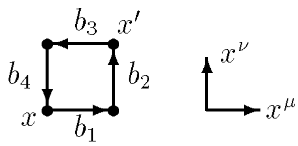


Figure 3: The field strength tensor

Determination of the gauge group from the initial conditions

We wish to explain that there will be a gauge group which can be determined from the initial state of the system at time $t = 0$, i.e. by the category $K_0 = K_{t=0}$. We emphasized that this initial state is to be considered as known. We will always think of the category as a communications network.

We will begin by discussing the issue for lattice gauge theory [22, 26, 25]. Then we point out that the construction generalizes readily to the general setup.

There are two ways of determining the gauge group from the initial conditions: as a holonomy group, and as the group of those invertible transformations which leave all gauge invariant observables invariant. Both are related because the "gauge invariant observables" are those which are invariant under parallel transport along all closed curves. Let us explain this in more detail

Holonomy group

Typically, lattice gauge theories live on hypercubic lattices $\Lambda = (a\mathbf{Z}^d)$ of lattice spacing a and dimension d. We are here interested in the Hamiltonian formulation where time remains continuous.

The sites of the lattice are denoted by x and the links from some site x to its nearest neighbour y by $b = (y, x)$. Associated with every site x is a complex vector space V_x. All these vector spaces are isomorphic, but they cannot be identified in a natural way because of the Naheinformationsprinzip.

There may be complex matter fields $\Psi(x) \in V_x$. In addition, a parallel transporter is associated with every link $b = (y, x)$. It is a invertible linear map

$$U(b) : V_x \mapsto V_y \ . \tag{26}$$

These paralleltransporters are the lattice gauge fields. They substitute for vector potentials $A = A_\mu dx^\mu$ in the continuum. The link $-b = (x, y)$ carries $U(-b) = U(b)^{-1}$.

This fits in the general framework as follows: Let X be the object at x. We denote the input space without the distinguished element $\emptyset$ by $A_X^\clubsuit$. Then

$$\begin{aligned} \Omega_X &= V_x = A_X^\clubsuit \ , & (27) \\ X\emptyset &= \Psi(x) \ , \quad X = id \ \text{ on } A_X^\clubsuit, & (28) \end{aligned}$$

and the elementary arrow attached to link b is $U(b)$. The composition of arrows is by composition of maps.

The gauge group is isomorphic to G^Λ for some compact group G which can act linearly on V_x. It consists of a collection of linear maps

$$g(x) : V_x \mapsto V_x \ , \quad (x \in \Lambda).$$

In the generic case, i.e. except for gauge field configurations which occur with probability zero, the gauge group is characterized by the condition

$$g(x) \in H(x) \ ,$$

where $H(x)$ is the group of all parallel transporters $U(C)$ along closed paths C from x to x. It is called the holonomy group. Groups $H(x)$ for different x are isomorphic.

This generalizes to the general situation as follows. An arrow $f \in Mor(Y, X)$ is called invertible if there exists an arrow $f^{-1} \in Mor(X, Y)$ such that

$$f \underset{X}{\circ} f^{-1} = \iota_Y \ \text{ and } \ f^{-1} \underset{Y}{\circ} f = \iota_X \ .$$

Let X be the object at x. The holonomy semigroup at x is $Mor(X, X)$. This is a semigroup because arrows can be composed. If we want a group, attention may be restricted to invertible arrows in $Mor(X, X)$. The gauge (semi)group is $\{Mor(X, X)\}|_{X \in Obj(K_0)}$. Under extra conditions, the holonomy groups are isomorphic for different x. In particular, this is true if all arrows are invertible and if the category is connected in the sense that $Mor(X, Y)$ is never empty.

Invariants, coupling constants

Basically, invariants are quantities associated with objects X which are invariant under parallel transport f along closed curves, or under

$$\hat{f} = Xf : \Omega_X \mapsto \Omega_X \ ,$$

for arbitrary $f \in Mor(X, X)$. These invariants are determined by functors which map the category into an unfrustrated category.

In particular, coupling constants which determine the Hamiltonian are invariants of this kind. They should be real valued. In this way the possible dynamics can be classified by the functorial maps of the initial category K_0 at time 0 (or of representations of it) into unfrustrated categories whose objects are subsets of the reals.

The gauge (semi)group consists of all collections $\{g(X)\}_{X \in Obj(K_0)}$ of maps $g(x) : \Omega_X \mapsto \Omega_X$ which leave all these invariants invariant.

The invariant quantities can be constructed out of vectors $v \in \Omega_X$, or of arrows $f \in Mor(X, X)$. Details are given in Appendix C and illustrated on the example of lattice gauge theory.

In general relativity, the metric tensor is invariant under parallel transport. This is the basic invariant. It expresses the existence of ideal clocks whose speed does not depend on their previous history. They can be synchronized. In other words, there is consensus about the meaning of (eigen)time.

Similarly, an economy cannot exist unless consensus can be achieved about the result of arithmetic operations on integer numbers by a process of synchronization (schools).

Generally speaking, invariants are those quantities about which a consensus can be achieved among all agents of the same type by a consistent process of synchronization.

Apart from very special examples (topological field theories) [27] without a "true" dynamics, it does not seem to be possible to formulate dynamical laws unless there exists at least one nontrivial invariant.

6 Life processes as stabilized critical fluctuations

This year Boltzmann's 150^{th} birthday is celebrated. On his tombstone, his formula $S = k \ln W$ for the entropy is engraved. One of his major achievements was the H-theorem which asserts that entropy increases. The question poses itself: Do we live in spite of this, or just becauseΓ It seems that both is the case.

In spite of

Experience with Monte Carlo simulations of lattice field theories shows the following. Let τ be the number of time steps until a new, statistically independent state of the system is reached. If the dynamics is local, and if the system is critical then τ diverges in the limit of infinite volume (infinitely many agents).

In this sense, equilibrium is never reached. Frustration can enhance this effect. For instance, spin glasses are extremely hard to equilibrate.

Whether the system is critical depends on some conditions on the parameters which determine the equilibrium state (coupling constants). There exist examples where systems are always critical, or at least almost. Pure gauge theories in 4 dimensions are like that. This is P. Bak's "self organized criticality" [29].

Wilsons renormalization group furnishes criteria when this happens.

At a critical point there are critical fluctuations of arbitrary spatial extension. Typically they can also have arbitrary life time if the dynamics is local.

A well known example is the critical point (P_c, T_c) of a real gas. The critical fluctuations of the gases density give rise to critical opalescence.

In a system with self interactions, fluctuations can generate fluctuations.

The work reported here grew out of attempts to gain a deeper understanding of how to fight critical slowing down in computer simulations of lattice gauge theory [16]. If the configurations show too much "will to live" and resistance against equilibration, this is the death of the Monte Carlo Method.

Just because: Deterministic chaos and self organization

Self organisation is the formation of complex structures out of simpler structures step by step in such a way, that the persistence of these structures - either static or by reproduction - is favored.

It is claimed [7] that chaos in nature is not only everywhere but is actually indispensable for the emergence of structure. A possible explanation could go like this

$$\text{chaos} \mapsto \text{chance} \mapsto \text{entropy production} \mapsto \text{stabilization of composite objects}$$

If entropy is produced when two objects form a composite object, then the process is irreversible and redissociation into the original two objects is impossible or at least suppressed.

This mechanism is familiar from physical chemistry; one would like to carry it over to a universal theory.

Actually it is very difficult to make this idea precise. A proposal how to do it will be submitted in what follows.

Let us begin by recalling that chaotic behavior of a dynamical system means sensitive dependence on initial conditions, see figure 4. A small change of the initial state ξ_0 may lead to a large deviation after sufficiently long time. Typically it grows exponentially with time.

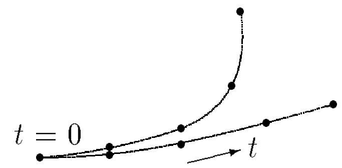


Figure 4: Chaos is sensitive dependence on initial conditions

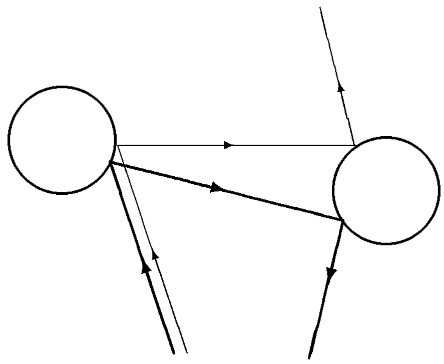


Figure 5: Sinai-Billard

Chaos in this sense combined with the limitations of a finite measuring accuracy for the initial state leads to unpredictability of the distant future. Examples in mechanics are well known

1. The 3-body problem was already studied by Poincaré [36, 33]

2. Scattering of atoms in a gas. According to Boltzmann, this leads to entropy production. In a simplified billiard model, one considers scattering off fixed solid disks in a plane . A small initial uncertainty in the impact parameter leads to a small uncertainty in the scattering angle. This gets converted into uncertainty of the position which grows with time. This leads to an uncertainty of the impact parameter for the next scattering, and so on. The uncertainty grows exponentially with the number of scatterings (figure 5). Sinai showed that this leads to a finite Kolmogorov entropy, thereby confirming Boltzmann's picture [35]. The diffusion process associated with this scattering has also been studied numerically [49, 38, 39].

3. Computer simulations indicate that the motion of the earths axis of rotation were chaotic if it were not stabilized by the moon. This would have affected the constancy of the change of seasons on earth and therefore the evolution of life [40]

We wish to consider unpredictable events as random. This leads to entropy according to Boltzmann's formula, $S = k \ln(\text{no. possibilities})$.

It is proposed to implement this idea as follows. Chance is introduced by adding an undetectable diffusion process to the Hamiltonian dynamics which affects only the stochastic components of the motion and tends to equilibrate them.

The diffusion may be thought of as a Brownian random motion which is superimposed on the Hamiltonian time development. This leads to a Langevin equation which is equivalent to a Fokker Planck equation for the distribution function $\rho(\xi, t)$ in phase space.

$$\frac{\partial}{\partial t}\rho + \{H, \rho\} - \frac{D}{2}\Delta^H \rho = 0 \ . \tag{29}$$

This involves a positive semidefinite second order differential operator $-D\Delta^H$ which is determined by the Hamiltonian dynamics and a metric - see below.

This is in the spirit of Prigogine's proposals concerning the origin of irreversibility [2].

There are two related problems which need to be solved in order to specify the diffusion process.

1. The picture of chaotic behavior shown in figure 4 presupposes definition of a distance. Because of covariance under canonical transformations, there is no preferred choice of coordinates and no natural metric on phase space which would define such a distance.
2. One needs to separate chaotic and nonchaotic components of the deviation $\zeta \in T_\xi\mathcal{M}$ from a given trajectory.

The diffusion process is supposed to be undetectable. This demand refers to a detector with a finite measuring accuracy. We are interested in the time development of subsystems under the influence of the environment. We think of this influence as a continuing measuring process such that the finite measuring accuracy is operational at all times. In other words, information which has once become undetectable does not reemerge again later on.

The metric on phase space which we will need embodies additional information beyond the Hamiltonian dynamics. It is interpreted as the correlation matrix for the measuring accuracies, $g^{\alpha\beta}(\xi) = < \Delta\xi^\alpha \Delta\xi^\beta >$.

It is assumed that this metric is consistent with the symplectic structure. If so, it can be used to separate the modes in the Jacobi equation for the deviation in a way which is stable under small perturbations of the dynamics. This separation singles out expanding and contracting directions in $T_\xi\mathcal{M}$. In a Hamiltonian dynamics, expanding and contracting directions always come in pairs. The diffusion process involves random forces which have components in the expanding directions and in the dual contracting directions, but not in the others.

Let us define a microscopic entropy

$$S(t) = \int_{\mathcal{M}} dv \, \rho(\xi, t) \ln \rho(\xi, t). \tag{30}$$

This quantity would be constant under a purely Hamiltonian time development [2]. But upon adding the diffusion process, this changes. The Fokker Planck equation of motion for the distribution function ρ implies

Theorem 7 (Second law of Thermo-Dynamics)

1. *The entropie cannot decrease with time,*

$$\frac{d}{dt}S(t) \geq 0. \tag{31}$$

2. *The entropy production $\frac{d}{dt}S$ can only vanish if the distribution function has vanishing directional derivatives $\zeta^\alpha\partial_\alpha\rho$ whenever $\zeta \in T_\xi\mathcal{M}_E$ is expanding or contracting. This is supposed to hold true for almost all ξ.*

The details will be given in the next section. $T_\xi\mathcal{M}_E$ consists of vectors which are tangent to the energy surface $H = E$.

Structure is in the eye of the beholder

According to our discussion of "things", structure is the formation of composite objects, of composite objects out of composite objects etc. But which collections of objects should be considered as composite objectsΓ

A definition of a "virtual composite object" will be proposed which specifies for every microscopic state $\xi_t \in \mathcal{M}$ which subcategories of the category K_t (of fundamental objects and their relations) can be regarded as composite objects.(They can overlap.) This defines the probability for the existence of some virtual composite object, given the distribution function ρ.

The definition depends not only on the Hamiltonian dynamics, but also on the metric. In other words, "*structure is in the eye of the beholder*". It depends on measuring accuracies.

In a gas, entropy depends on the number of degrees of freedom. In our situation we can count the number $d(\xi)$ of expanding directions in $T_\xi\mathcal{M}$. We call $d(\xi)$ the *dimension of instability.*

Given the category K which is determined by a point $\xi \in \mathcal{M}$ in phase space, let $O \subset Obj(K)$ be a subset of its objects, and consider the category $K(O)$ which is generated by these objects and the elementary relations among them.

The state of the objects and elementary relations in $K(O)$ is denoted by ξ_O. All other state variables ξ^β which describe the environment of $K(O)$ are considered fixed. Suppose that $O = \bigcup O_i$ is an arbitrary partition of O in disjoint nonempty subsets O_i. The state variables are classified accordingly. $\xi_O = \{\xi_{O_i}, \xi_{\partial O_i}\}$. Herein, $\{\xi_{\partial O_i}\}$ are variables which determine the state of arrows in $K(O)$ which connect objects in different subsets O_i. The categories $K(O)$ and $K(O_i)$ will have dimensions of instability $d(\xi_O)$ and $d(\xi_{O_i})$, respectively.

Definition 8 (virtual composite object) $K(O)$ *is a virtual composite object if the following strict inequality on dimensions of instability is fulfilled for every nontrivial partition* $O = \bigcup O_i$

$$d(\xi_O) > \sum_i d(\xi_{O_i})$$

In words: The virtual composite object exists if it can be stabilized by entropy production. Its microscopic state is ξ_O, and its time development is given by eq.(131) of Appendix D.

Let us note that $K(O)$ cannot exist as a virtual composite object if it is not connected as a category, i.e. if it could be decomposed into two subcategories with no elementary relations between.

Stability properties of the dimension of instability under perturbations are discussed in the next section.

7 Construction of the diffusion process

The evolution in time of a deviation

$$\delta\xi_t = \epsilon\zeta(t) \quad , \quad \zeta(t) \in T_{\xi_t}\mathcal{M}$$

along a trajectory $t \mapsto \xi_t$ is described by the Jacobi-Equation

$$\frac{d}{dt}\zeta^\alpha(t) = N^\alpha_\gamma(t)\zeta^\gamma(t) \tag{32}$$

$$N^\alpha_\gamma(t) = -\partial_\gamma(\omega^{\alpha\beta}\partial_\beta H) \; . \tag{33}$$

The solution furnishes a map from one space to another

$$\phi^*_{t0} : T_{\xi_0}\mathcal{M} \mapsto T_{\xi_t}\mathcal{M}. \tag{34}$$

It makes no sense to speak of eigenvalues of such a map in general.

For this reason, the standard definitions of Liapunov exponents [34] assume a critical point or periodic motion so that $\xi_t = \xi_0$. In this case, ϕ^*_{t0} is a symplectic map of $T_{\xi_0}\mathcal{M}$ whose eigenvalues can be classified in the standard fashion. Those eigenvalues which do not lie on the unit circle come in quadruples $\lambda, \bar{\lambda}, \lambda^{-1}, \bar{\lambda}^{-1}$, or in pairs, if λ is real.

If we want to formulate a local thermo-dynamic equation of motion, and if we insist on our intention of accepting only the unstable (chaotic) modes of the motion as a source of stochasticity, then we need a local stability criterion. It should not involve prevoyance of the distant future, either.

Krein's classic stability theory relies essentially on the behaviour of the eigenvalues of a symplectic matrix under small perturbations. In particular, an eigenvalue $\neq 1$ can leave the unit circle only if it collides with another eigenvalue. To preserve this stability property, we want to interpret the map (34) as a symplectic map. This is only possible if we can identify the two spaces in a natural way.

Such an identification is possible with the help of a metric connection on $\mathcal{M}$ which is compatible with the symplectic structure.

With its help, vectors $\zeta \in T_{\xi_t}\mathcal{M}$ can be parallel transported along the trajectory back into T_{ξ_0}. The compatibility of the metric connection with the symplectic structure will ensure that the resulting map

$$\phi^\Gamma_{t0} : T_{\xi_0} \mapsto T_{\xi_0}. \tag{35}$$

is symplectic.

A metric connection is given by

1. a Riemannian metric on $\mathcal{M}$, i.e. by a (positive) scalar product $<,>_\xi$ in $T_\xi\mathcal{M}$,

$$< \zeta, \eta >_\xi = \zeta^\alpha g_{\alpha\beta}(\xi)\eta^\beta \ . \tag{36}$$

2. a connection which is compatible with the metric. In a coordinate basis it is given by connection coefficients ${\Gamma^\alpha}_{\beta\gamma}(\xi)$ Compatibility with the metric means that the scalar product is invariant under parallel transport, or, equivalently, that the metric tensor is covariantly constant, ${g^{\alpha\beta}}_{;\gamma} = 0$.

The inverse metric tensor is denoted by $g^{\alpha\beta}$. The connection may have a nonvanishing torsion ${S^\alpha}_{\beta\gamma}(\xi) = \frac{1}{2}\left({\Gamma^\alpha}_{\beta\gamma}(\xi) - {\Gamma^\alpha}_{\gamma\beta}(\xi)\right)$. We use the customary semicolon notation for covariant derivatives, e.g.

$${\omega^{\alpha\beta}}_{;\gamma} = \partial_\gamma \omega^{\alpha\beta} + {\Gamma^\alpha}_{\delta\gamma}\omega^{\delta\beta} + {\Gamma^\beta}_{\delta\gamma}\omega^{\alpha\delta} \ . \tag{37}$$

and similarly for the inverse metric tensor.

It suffices to consider infinitesimal t. The time derivative of ϕ^Γ_{t0} equals the covariant time derivative of ϕ^*_{t0} and is described by the covariant Jacobi matrix K.

$$\begin{aligned} \frac{D}{dt}\zeta^\alpha(t) &= {K^\alpha}_\beta(\xi_t)\zeta^\beta(t) && (38) \\ {K^\alpha}_\gamma(\xi) &= -\partial_\gamma(\omega^{\alpha\beta}\partial_\beta H) \ - \omega^{\beta\delta}\partial_\delta H {\Gamma^\alpha}_{\gamma\beta} \ . && (39) \end{aligned}$$

K describes the time evolution of a deviation in a comoving basis. [4] It transforms covariantly under coordinate transformations and also under transition to an anholonomic basis.

[4] By comoving we mean autoparallel along the trajectory

Definition 9 Compatibility of the symplectic structure with a metric connection *holds true if the volume form in local coordinates has the form* $dv = \sqrt{g}\prod d\xi^\alpha$ *and if*

$$\omega^{\alpha\epsilon}{}_{;\beta}\omega^{\beta\gamma} - \omega^{\gamma\epsilon}{}_{;\beta}\omega^{\beta\alpha} = 2\{S^{\epsilon}{}_{\beta\delta}\omega^{\alpha\beta}\omega^{\delta\gamma} - S^{\alpha}{}_{\beta\delta}\omega^{\delta\epsilon}\omega^{\beta\gamma} - S^{\gamma}{}_{\beta\delta}\omega^{\delta\epsilon}\omega^{\alpha\beta}\} \ . \tag{40}$$

A metric is called compatible with the symplectic structure if there exists a corresponding metric connection which is.

Let us note that the condition is satisfied if the symplectic matrix is covariantly constant and the torsion vanishes. Using (40), one verifies that the covariant Jacobi matrix is an element of a symplectic Lie algebra,

$$K^{\alpha}{}_{\beta}(\xi)\omega^{\beta\gamma}(\xi) + K^{\gamma}{}_{\delta}\omega^{\alpha\delta}(\xi) = 0 \ . \tag{41}$$

Therefore, the time evolution operator ϕ_t^* in a comoving basis is described by a symplectic matrix as desired. The covariant Jacobi matrix depends on the connection. But it will turn out that the diffusion process which we will construct depends only on the metric, at least if K is diagonalizable.

Example 10 *Euclidean phase space* $\mathcal{M} = \mathbf{R}^{2n} = \{(p,q)\}$ *with the usual Euclidean metric and with Poisson brackets* $\{p_i, q^i\} = \delta_i^j$.

In a Hamiltonian dynamics with an invertible symplectic matrix one can always choose local coordinates as in the example because of Darboux theorem. This shows that the condition can always be fulfilled locally. This is enough to define the diffusion process

Given the symplectic structure, there is in general much freedom to choose a metric. In the example, the freedom of transformations $p \mapsto \lambda p,\ q \mapsto \lambda^{-1}$ remains, for instance. In the reverse direction it is different [37]. This suggests to consider the metric as the basic quantity. If we regard the detectors as part of the system, then the determination of the metric from initial conditions will have to invoke the same principles as in the case of the Hamiltonian.

It follows from the symplecticity of ϕ_t^Γ occur only in singlets, pairs and quadruples as follows [32]

1. singlet 0;
2. pairs $\pm\lambda$ (λ real);
3. pairs $\lambda, \overline{\lambda}$ (λ imaginary):
4. quadruples $\pm\lambda, \pm\overline{\lambda}$ (λ complex).

Eigenvectors to eigenvalues λ with $\Re\lambda \neq 0$ will be called *hyperbolic* . They are *expanding* if $\Re\lambda > 0$ and *contracting* if $\Re\lambda < 0$; Eigenvectors with $\Re\lambda = 0, \lambda \neq 0$ are called *elliptic*. (If K is not diagonalizable, there can also be parabolic vectors. They are associated with a spectral value 0 of K. Compare Appendix D)

For hyperbolic eigenvectors with $Im\lambda \neq 0$ the expansion or contraction is associated with a rotation. We may imagine that the time evolution is discretized in such a way that expansion/contraction is alternating with rotations. We want to associate a diffusion process only with the expansion and contraction. As a simple illustration, a two dimensional example is discussed in Appendix D. In contrast with this example we shall wish to constrain the diffusion process to the energy surface, however.

Definition 11 (Modulus of the covariant Jacobi matrix, diffusion operator) *Suppose a Hamiltonian dynamics is specified together with a metric connection on $\mathcal{M}$ which is compatible with the symplectic structure, with metric tensor $D^{-1}g_{\alpha\beta}(\xi)$.* [5] *We denote the energy surface $H = E$ by $\mathcal{M}_E$.* [6] *Let $\Pi_E(\xi) : T_\xi\mathcal{M} \mapsto T_\xi\mathcal{M}_E$ the projector onto vectors which are tangential to the energy plane, and let $\Pi_i(\xi)$ the projectors which project onto real linear combinations of eigenvectors of the covariant Jacobi matrix $K(\xi)$ to eigenvalues $(\lambda, \bar{\lambda})$ with $\Re\lambda \neq 0$. We define th* modulus of the Jacobi matrix

$$|K(\xi)|^\alpha{}_\beta = \sum_i |\Re\lambda_i|\, \Pi_i(\xi)^\alpha{}_\beta\ . \tag{42}$$

and its restriction to directions tangential to the energy surface

$$|K_E| = \Pi_E |K| \Pi_E\ . \tag{43}$$

The diffusion operator associated with H is the following positive semidefinite differential operator of 2^{nd} order on $\mathcal{M}$, multiplied with D.

$$-\Delta^H = \frac{1}{\sqrt{g}} \partial_\alpha \sqrt{g} |K_E|^\alpha{}_\beta g^{\beta\gamma} \partial_\gamma\ . \tag{44}$$

Arbitrary vectors $\zeta \in T_\xi\mathcal{M}$ are called hyperbolic, *if $\zeta \in$ range $|K|$.*

Proposition 12 (Independence of the diffusion process of the choice of connection) *If K is diagonalizable, then $|K|$, and therefore also $|K_E|$ and Δ^H depend on the metric, which is assumed compatible with the symplectic structure, but they are independent of the choice of the metric connection.*

PROOF: Consider the matrix ϕ^*_{t0} of the time evolution from $t = 0$ to t. Let the comoving basis be fixed at $t = 0$, we choose it orthonormal. When the metric connection is changed, but not the metric, then the new comoving frame at ξ_t is obtained from the old one by a rotation $V_t = V_t^{*-1}$. Therefore

$$\phi^*_{t0} \mapsto V^*_t \phi^*_{t0}.$$

Consequently,

$$|\phi^*_{t0}|^2 = (\phi^*_{t0})^* \phi^*_{t0}$$

remains invariant. For small t we have $\phi^*_{t0} = 1 + tK(\xi_0)$. Let $K(\xi_0) = \sum \lambda_i \Pi_i$. In an orthonormal basis, the projection matrices Π_i are hermitian. Therefore

$$|\phi^*_{t0}|^2 = \sum e^{2t\Re\lambda_i} \Pi_i.$$

It follows that

$$\frac{1}{t}\left(\frac{1}{2}\phi^*_{t0}|^2 + \frac{1}{2}|\phi^*_{t0}|^{-2} - 1\right)^{\frac{1}{2}} = |K|(\xi_0) + O(t)\ . \tag{45}$$

Therefore $|K|$ is also invariant. q.e.d.

In the parabolic case, K is triangular. This case was excluded by the assumption of diagonalizability of K. We remark that the choice of eq.(45) as the general definition of $|K|$ would lead to a diffusion process also for parabolic modes. We take the attitude that diffusion processes for parabolic modes belong to the realm of quantum mechanics, since they are associated with measuring uncertainties with $< \Delta p_i \Delta q^i > \neq 0$.

[5] We extract a factor D in order to be able to normalize the metric tensor $g_{\alpha\beta}$ by convention.

[6] In general relativistic theories $E = 0$ is often enforced by other constraints

Corollary 13 $|K|(\xi)$ *maps the part* $T_\xi \mathcal{M}_E$ *of* $T_\xi \mathcal{M}$ *which is tangential to the energy surface, to itself.*

PROOF: Because of energy conservation, the time evolution of a deviation maps vectors which are tangential to the energy surface $\mathcal{M}_E$ into vectors with the same property. The same is true of the parallel transport of tangent vectors if we use the Riemannian connection on $\mathcal{M}_E$ which is furnished by the restriction of the metric to $\mathcal{M}_E$. In this way, $|K|(\xi_0)$ is defined as a map of $T_{\xi_0}\mathcal{M}_E$ to itself. q.e.d.

With these definitions, the Fokker Planck equation of motion (29) for the distribution function can now be written down. The decisive difference compared to familiar versions of a Fokker Planck equation as used e.g. in [3] consists in the fact that only the chaotic modes are sources of a diffusion process. If there are only stable (elliptic) modes, then the eigenvalues of the covariant Jacobi matrix are imaginary and $\Delta^H = 0$.

Let us turn to a discussion of stability properties of the dimension of instability. By construction, $K(\xi)$ is a symplectic matrix. If ξ changes, or if a perturbation is added, then a new expanding direction, and therefore a rise of the dimension of instability can only arise out of an eigenvalue 0 or if two eigenvalues collide on the circle. (According to Krein's theory they must in addition have "opposite sign", cp. [32])

This consideration implies in particular stability properties against infinitesimal changes of the Riemannian metric.

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Appendix A: Proof of the representation theorem 3 for categories

Given a category K, we write $Mor(Y,*)$ for the set of all its arrows to Y etc. We define

$$In(Y) = Mor(Y,*) \quad , \quad Out(Y) = Mor(*,Y) \; .$$

We write $X = \alpha(f)$ if $f \in Mor(Y,X) \subset In(Y)$, and correspondingly $Z = \omega(f)$ if $f \in Mor(Z,Y) \subset Out(Y)$. The output space will be defined as a subspace Ω_Y of Ω_Y^{virt}. Ω_Y^{virt} consists of maps

$$\zeta : Out_Y \mapsto Mor(*,*)$$

with the property $\zeta(f) \in Mor(\omega(f),*)$.

An object Y will act as a map

$$Y : In(Y) \mapsto \Omega_Y .$$

according to

$$Yf(g) = g \underset{Y}{\circ} f \quad (g \in Out(Y)).$$

The output space is defined as the image of Y, and the input space as space of equivalence classes (if necessary) of elements of $In_K(Y)$, which Y maps into the same $\zeta \in \Omega_Y^{virt}$.

$$\Omega_Y = IM\ Y \subset \Omega_Y^{virt} , \tag{46}$$
$$A_Y = In(Y)/KER\ Y . \tag{47}$$

Y is invertible as a map from A_Y to Ω_Y. Its inverse is ι_Y. The empty input $\emptyset \in A_Y$ is defined as the equivalence class of $\iota_Y \in Mor(Y, Y) \subset In(Y)$.

An arrow $f \in Mor(Y, X)$ is defined as a map $\Omega_X \mapsto A_Y$ by use use of the map $\iota_X : \Omega_Y \mapsto A_Y$, as follows.

$$f = \hat{f} \underset{X}{\circ} \iota_X , \tag{48}$$
$$\hat{f}(g) = f \underset{X}{\circ} g \quad \text{for } g \in Mor(X, *) . \tag{49}$$

The last formula defines $\hat{f}$ as a map from $In(X)$ to $In(Y)$. This map passes to equivalence classes (47) thereby defining a map $A_X \mapsto A_Y$. The composition rule (21)holds. q.e.d.

8 Appendix B: Lattice models

Geometry of a hypercubic lattice

We consider a simplicial complex [50] or, more generally a cell complex. A hypercubic lattice is the simplest and most interesting case. But the following considerations can be generalized.

The lattice of dimension ν consists of 0-cells (sites), 1-cells (links), 2-cells (plaquettes = squares), possibly 3-cells (cubes) etc.

Let C^n the set of all n-cells. The cells are oriented for $n \neq 0$. The cell with the opposite orientation to c is denoted $-c$. Formal sums

$$C = \sum_{c \in C^n} a_c c \tag{50}$$

of n-cells c with integer coefficients are called n-chains. The boundary of $(n+1)$-cell a n-cell with coefficients $0, \pm 1$. It is determined by the incidence matrix ω.

$$\partial c = \sum_{e \in C^n} \omega^{ce} e \tag{51}$$

$\omega^{cb} = 1,(-1)$ if b is in the boundary of c, and if it has the same (opposite) orientation. For instance, a link b from site x to site y has boundary $\partial b = y - x$. The boundary ∂p of a plaquette p is the sum of four oriented links, etc.

We stipulate that $\omega^{ce} = 0$ if c is a n-cell and e is a k-cell with $k \neq n \pm 1$, and

$$\omega^{ec} = -\omega^{ce} . \tag{52}$$

In this chapter we want to consider models of a hamiltonian dynamics in which the incidence matrix plays the role of the symplectic matrix ω. The boundary of a boundary vanishes.

$$\partial^2 = 0. \tag{53}$$

Besides the boundary operator ∂ we will also need the coboundary operator ∂^*. The coboundary $\partial^* e$ of a $(n-1)$-cell e is a n-chain. It is defined by

$$\partial^* e = \sum_{c \in C^n} \omega^{ce} c \ . \tag{54}$$

The dynamical variables of our models will be functions f which assign a real number $f(c)$ to every cell c of some given dimension n. The definition of f can be extended to chains of the form (50) by setting $f(C) = \sum_{c \in C^n} a_c f(c)$. f becomes a $\mathbf{Z}$-linear map from n-chains to $\mathbf{R}$. Such functions are called *n-cochains*. The exterior derivative d and the coderivative d^* act on cochains. They are defined by

$$df(c) = f(\partial c) \ = \ \sum_{e \in C^{n-1}} \omega^{ce} f(e) \ . \tag{55}$$

$$d^* g(c) = g(\partial^* c) \ = \ \sum_{e \in C^{n+1}} \omega^{ec} g(e) \tag{56}$$

for a n-chain c, $(n-1)$-cochain f and $(n+1)$-cochain g. They obey $d^2 = 0 = d^{*2}$. d^* ist the adjoint operator of d in a space of square summable cochains.

We will also write f_c in place of $f(c)$.

A model with frustration

In this model, the objects are labelled by the sites x of a ν-dimensional hypercubic lattice. Their states are given by real variables $\pi(x)$. The elementary arrows are assigned to links b. Their state is described by real variables $\sigma(b) = -\sigma(-b)$

The symplectic matrix shall be given by the incidence matrix. As a Hamiltonian we adopt

$$H = -\frac{1}{2}\{\sum_x \pi(x)\pi(-x) + \sum_b \sigma(b)\sigma(-b)\} \ . \tag{57}$$

The sum runs over all sites x und all links b. Links which differ only in orientation are not counted twice. $\pi(-x)$ is to be read as $-\pi(x)$.

The Poisson-brackets are

$$\{\sigma(b), \pi(x)\} = \omega^{bx} \ . \tag{58}$$

The quantities $d\sigma(p)$ and $d^*\pi(b)$ are degeneracy invariants. They are therefore determined by the initial conditions. Let us demonstrate it for $d\sigma$. Its only possibly nonvanishing Poisson bracket is

$$\{d\sigma(p), \pi(x)\} \ = \ \sum_b \omega^{pb} \{\sigma(b), \pi(x)\} \tag{59}$$

$$= \ \sum_b \omega^{pb} \omega^{bx} = 0 \tag{60}$$

because $\partial^2 = 0$.

We consider the case that the initial conditions fix the values

$$d\sigma(p) \ = \ \rho(p), \tag{61}$$

$$d^*\pi \ = \ 0 \tag{62}$$

A similar situation arises in electrodynamics with electric charges or magnetic monopoles (Both cases are related by a duality transformation).

The Hamilton equations of motion are

$$\begin{aligned} \frac{d}{dt}\pi(x) &= d^*\sigma(x) && (63)\\ \frac{d}{dt}\sigma(b) &= -d\pi(b) && (64) \end{aligned}$$

The composition of arrows shall be effected by addition of variables $\sigma(b)$ along the path. The identity arrow is represented by addition of 0.

$\rho(p)$ determines whether the category is frustrated or not. Let $b_1 \circ b_1'$ and $b_2 \circ b_2'$ the two paths between diametrically opposite corners x and y of a plaquette p, and let f_1 and f_2 the corresponding arrows $f_i = \sigma(b_i) + \sigma(b_i')$. Depending on the orientation of the plaquette

$$f_2^{-1} \circ f_1 = \pm\rho(p) \tag{65}$$

The category is frustrated if $f_2^{-1} \circ f_1 \neq \iota_x$ for at least one p, i.e. if at least one $\rho(p) \neq 0$. In this model the frustration $\rho(p)$ is a constant of motion.

Let us now try to solve the constraint (61).

$$\sigma(b) = \tilde{\sigma}(b) + d\phi(b) \tag{66}$$

Herein $\tilde{\sigma}$ is a particular solution. It can be considered given by ρ and possibly a cohomology class. The Poisson brackets are reproduced by

$$\{\pi(x), \phi(y)\} = \delta_{xy}.$$

ϕ is a function on sites. Assuming the lattice has no boundary, we can integrate (sum) by parts, and the Hamiltonian assumes the following form

$$H = -\frac{1}{2}\sum_x [\pi(x)\pi(-x) + \phi(x)\Delta\phi(x)] + \sum_b \tilde{\sigma}(b)\phi(\partial b) + const\ . \tag{67}$$

In this formula $-\Delta = d^*d$ is the Laplacian acting on functions on sites of the lattice.

One verifies that $d^*\pi$ is no longer a degeneracy invariant. Now it generates gauge transformations of ϕ

$$\delta\phi(x) \equiv \sum_b \lambda(b)\{d^*\pi(b), \phi(x)\} = -d^*\lambda(x) \tag{68}$$

The gauge function λ lives on links b.

This formulation has the draw back that the Hamiltonian has become complicated and contains quantities which were furnished by he initial condition in the original formulation.

A better alternative is a first order formalism where σ is retained as a field besides ϕ, and the original Hamiltonian is retained. There is now again a degeneracy invariant

$$\sigma(b) - d\phi(b) \tag{69}$$

which is fixed by the initial conditions. This formalism generalizes to nonabelian gauge theories.

Appendix C: Invariants and types of objects

We consider first lattice gauge theory as an example

In lattice gauge theory a vector space V_x of some dimension N is attached to every site x. Linear maps

$$\mathcal{U}(C) : V_x \mapsto V_y \tag{70}$$

are associated with paths C on the lattice from x to y. They are known as parallel transporters. Compare eq.(26)ff.

The gauge group may be fixed by equipping the vector spaces with additional structure which is required to be invariant under parallel transport [18].

An invariant scalar product $<,>$ is the most important example.

$$\begin{aligned} <,>: V_x \times V_x &\mapsto \mathbf{C} \qquad (71)\\ (v,w) &\mapsto <v,w>_x \qquad (72)\end{aligned}$$

with the invariance property

$$<v,w>_x = <\mathcal{U}(C)v, \mathcal{U}(C)w>_y \quad . \tag{73}$$

This leaves the freedom of gauge transformations which leave the scalar product invariant

$$\begin{aligned} \mathcal{S}(x) : V_x &\mapsto V_x \ , \qquad (74)\\ <\mathcal{S}(x)v, \mathcal{S}(x)w>_x &= <v,w>_x \ , \qquad (75)\\ \mathcal{U}(C) &\mapsto \mathcal{S}(y)\mathcal{U}(C)\mathcal{S}(x)^{-1} \qquad (76)\end{aligned}$$

The group of such gauge transformations is isomorphic to $U(N)$.

Starting from some scalar product at a fixed site $\hat{x}$ an invariant scalar product can be defined for all x, if the invariance property (73) holds for *closed* paths C which begin and terminate at $x = y = \hat{x}$.

The gauge group is further reduced to $SU(N)$ if the existence of a determinant is postulated which is invariant under parallel transport.

$$\begin{aligned} det_x : V_x \times ... \times V_x &\mapsto \mathbf{C} \ , \quad \text{(totally antisymmetric)} \qquad (77)\\ (v_1, ... v_N) &\mapsto det_x(v_1 \wedge ... \wedge v_N) \qquad (78)\\ det_x(v_1 \wedge ... \wedge v_N) &= det_y(\mathcal{U}(C)v_1 \wedge ... \wedge \mathcal{U}(C)v_N) \ . \qquad (79)\end{aligned}$$

Let us mention that the linearity of the maps (70), the construction of tensor products of spaces $W_x = V_x \otimes ... \otimes V_x$ of V_x, and the rules for parallel transport of vector products in these tensor product spaces W_x can all be interpreted as existence of invariants which are mapped into themselves under parallel transport along closed paths. If the tensor product is changed in a suitable way [15], one arrives at quantum group gauge theories [17]

We want to generalize this construction to arbitrary categories K. We use their representation as communication networks.

Definition 14 (Invariants) *An invariant is a functor which maps K or a category which is derived from K to an unfrustrated category k.*

The invariant is called real valued *or a* coupling constant, *if the objects in k are subsets of* $\mathbf{R}$.

We are interested in invariants which are defined for every category in some family. In order to preserve the locality properties, inclusions should be respected. Therefore we demand that the functor on a subcategory should be defined by restriction.

Example 15 *: An* object-invariant *F of degree n is defined by a partition of objects X in equivalence classes $[X]$, called types of objects, and maps*

$$F_X : \Omega_X \times ...\Omega_X \mapsto M_{[X]} \quad (n \ factors) \tag{80}$$

of n-fold products of output spaces in spaces $M_{[X]}$. F_X is defined for every object X and has the following property. If there is an arrow from X to Y, then a map with the following properties is defined

$$\phi_{[Y][X]} : M_{[X]} \mapsto M_{[Y]} \ , \tag{81}$$

$$\phi_{[X][X]} = id \quad , \quad \phi_{[Z][Y]}\phi_{[Y][X]} = \phi_{[Z][X]} \ , \tag{82}$$

and for every arrow $f : X \mapsto Y$

$$\phi_{[Y][X]} F_X(v_1, ..., v_n) = F_Y(Yfv_1, ..., Yfv_n) \ . \tag{83}$$

The defining properties of an object-invariant of degree n make it a functor. The functor maps the product category K^n [7] into an unfrustrated category with objects $M_{[X]}$ and arrows $\phi_{[Y][X]}$. The composition of arrows in the image category is defined as the composition of maps. Since the arrows are maps, the image category is a communication network.

The special case with only one type $M_{[X]}$ is especially important.

The following type of invariant is similar to the object invariants.

By assumption, the spaces Ω_X are differentiable manifolds. Therefore they possess tangent spaces $T\Omega_X$. The elements of $T_v\Omega_X$ are tangent vectors to curves through $v \in \Omega_X$. Given $f : X \mapsto Y$, then $Yf : \Omega_X \mapsto \Omega_Y$ is defined as a map, and therefore also the map of curves in Ω_X into curves in Ω_Y. By canonical construction, this defines a map

$$(Yf)^* : T\Omega_X \mapsto T\Omega_Y \tag{84}$$

Example 16 (tangential invariant) *A tangential invariant is defined by a partition of the objects X in equivalence classes $[X]$, and maps*

$$L_X : T\Omega_X \times T\Omega_X \mapsto M_{[X]} \tag{85}$$

in spaces $M_{[X]}$. L_X is defined for every object and has the same properties as the map F_X for an object invariant of 2^{nd} degree, except that the map Yf must be replaced by $(Yf)^*$.

The tangential invariants can be regarded as functors on a tangential category TK derived from K.

Finally there exist invariants for arrows $f \in Mor(X, X)$.

Example 17 *:* (loop invariant) *A loop invariant G is defined by a partition of the objects in types $[X]$ and maps*

$$G_X : Mor(X, X) \mapsto M_{[X]} \tag{86}$$

[7] The input spaces in K^n are $A_X \times ... \times A_X$ (n factors) etc. The action of objects and arrows is declared in the natural way

in spaces $M_{[X]}$ with the following properties

$$\phi_{[Y][X]} : M_{[X]} \mapsto M_{[Y]} \ , \tag{87}$$

$$\phi_{[X][X]} = id \quad , \quad \phi_{[Z][Y]}\phi_{[Y][X]} = \phi_{[Z][X]} \tag{88}$$

such that for every arrow $g : X \mapsto Y$ which possesses an inverse $g^{-1} : Y \mapsto X$

$$\phi_{[Y][X]} G_X(f) = G_Y(g \underset{X}{\circ} f \underset{X}{\circ} g^{-1}) \ . \tag{89}$$

for every $f \in Mor(X, X)$.

The defining properties of a loop invariant make it into a functor of a category K' into an unfrustrated category. K' is the category which is obtained from K as follows. Its arrows are arrows connecting different objects of K and the identity arrows, and its objects are arrows $s \in Mor(X, X)$ of K. $f \underset{s}{\circ} g$ ist defined by composition of arrows in K.

We give an example of a loop invariant. Let A_X and Ω_X real vector spaces on which arrows and objects act as linear maps. Then the trace of the map $Xf : \Omega_X \mapsto \Omega_X$ associated with loops $f \in Mor(X, X)$ is a loop invariant.

By definition, the possibility of dividing in types is determined by the existing functorial maps into unfrustrated categories.

Intuitively this means that the communication network admits messages of an agent about his type. The message received by the recipient does not depend on the path which the message took. (Messages for which this is not the case might be called "gossip").

Let us now discuss how Hamiltonians are constructed.

Hamiltonians may be constructed out of invariants which do not depend on a choice of coordinates. They will then obey the relativity principle.

Let b an arbitrary link and let $f : X \mapsto Y$ the corresponding elementary arrow. Its state will be determined by a point $\xi \in \mathcal{M}$ as will be the states of the objects. Let F^a object invariants of degree 2. We write $\Psi_X = X\emptyset$ as before. Set

$$H_b(\xi) = \sum_a F^a_Y(Yf\Psi_X, \Psi_Y) \ . \tag{90}$$

This fulfills our locality postulates. We agree to include here also selflinks $s = (x, x)$ with which the arrow $f\iota_X$ is associated. In this case we get

$$H_x(\xi) = \sum F^a_X(\Psi_X, \Psi_X) \ . \tag{91}$$

Sometimes object invariant F^a of degree 1 exist. They also furnish contributions

$$H_x(\xi) = \sum F^a_X(\Psi_X) \tag{92}$$

to the Hamiltonian, and the same is true of invariants of higher order.

In the case of selflinks s which connect a node x with itself, there exist additional possible contributions to H. They are constructed from loop invariants. Let $f : X \mapsto X$ the arrow associated with s. Set

$$H_s(\xi) = \sum_a G_a(f) \ . \tag{93}$$

In addition to the Hamiltonian one needs a symplectic matrix $\omega(\xi)$. Its pseudo-inverse $\Omega(\xi)$ is an antisymmetric bilinear form

$$T_\xi\mathcal{M} \times T_\xi\mathcal{M} \mapsto \mathbf{R} \tag{94}$$

Such bilinear forms can be made out of tangential invariants.

For the purpose of formulating a Thermo-dynamics in the sense of Prigogine, a metric on $\mathcal{M}$ was needed. The metric tensor defines also a bilinear form (94), it must be symmetric and positive definite.

Appendix D: Dynamics and diffusion in a 2-dimensional phase space

It is very instructive to consider the simplest possible example: time independent linear equations of motion in a 2-dimensional phase space $\mathcal{M} = \mathbf{R}^2$. The coordinates in $\mathcal{M}$ shall be denoted by $\xi = (\xi^1, \xi^2)^t$ (column vector). The symplectic form shall be $\Omega = 2d\xi^1 \wedge d\xi^2$, hence $\omega^{21} = -\omega^{12} = 1$.

The possible equations of motion and their solutions are given by 1-parameter groups of symplectic transformations. They are of the form

$$\frac{d}{dt}\xi(t) = X\xi(t), \tag{95}$$

$$\xi(t) = \phi_t\xi(0)\ ,\ \phi_t \in SL(2,\mathbf{R}) \tag{96}$$

$$\phi_t = \exp Xt \quad ,\ X \in sl(2,\mathbf{R})\ , \tag{97}$$

The generator X may be an elliptic, parabolic or hyperbolic element of $sl(2,\mathbf{R})$ with eigenvalues $\pm i\alpha$, 0 or $\pm\alpha$, $\alpha > 0$, respectively.

An unstable direction arises only in the hyperbolic case. This case will be considered first.

8.1 Hyperbolic motion

X can be diagonalized by real basis transformations. The Hamiltonian will then take the form

$$H = \alpha\xi^1\xi^2 \quad (\alpha > 0)\ . \tag{98}$$

The Hamiltonian equations of motion read

$$\frac{d}{dt}\xi^1(t) = \alpha\xi^1(t)\ ,\ \frac{d}{dt}\xi^2(t) = -\alpha\xi^2(t)\ , \tag{99}$$

with solution

$$\xi^1(t) = e^{+\alpha t}\xi^1(0)\ ,\ \xi^2(t) = e^{-\alpha t}\xi^1(0)\ , \tag{100}$$

The point 0 is a critical point. The 1-direction is expanding, the 2-direction contracting, with Liapunov exponent $\pm\alpha$.

The equation of motion is already linear and is therefore identical to its linearization. The Jacobi matrix

$$K = diag(\alpha, -\alpha)\ . \tag{101}$$

The Hamiltonian equations of motion for the normalized distribution function read (Notation: $\partial_i = \partial/\partial\xi^i$)

$$\frac{d}{dt}\rho + \alpha(\xi^1\partial_1 - \xi^2\partial_2)\rho = 0 \tag{102}$$

The critical point 0 furnishes the solution

$$\rho(\xi,t)=\delta(\xi^1)\delta(\xi^2). \tag{103}$$

If the distribution is Gaussian at time 0, then it remains Gaussian at all times.

$$\begin{aligned}
\rho(\xi,t) &= \rho_0^1(\xi^1,t)\rho_0^2(\xi^2,t), && (104)\\
\rho_0^1(\xi^1,t) &= (2\pi f_0^1(t))^{-\frac{1}{2}}\exp[-(\xi^1)^2/2f_0^1(t)] \quad , \quad f_0^1(t)=\lambda_1 e^{2\alpha t}\ , && (105)\\
\rho_0^2(\xi^1,t) &= (2\pi f_0^2(t))^{-\frac{1}{2}}\exp[-(\xi^2)^2/2f_0^2(t)] \quad , \quad f_0^2(t)=\lambda_2 e^{-2\alpha t}\ . && (106)
\end{aligned}$$

The parameter λ_i determine the width of the distribution at time 0.

It is seen that the Gauss distribution in ξ^1-direction expands exponentially with time t,

$$\Delta\xi^1=e^{\alpha t}\ , \tag{107}$$

while exponential contraction is observed in ξ^2-direction. The microscopic entropy

$$S=-\int d\xi^1 d\xi^2 \rho(\xi,t)\ln\rho(\xi,t)=\frac{1}{2}\sum_i[\ln 2\pi\lambda_i(\Delta\xi^i)^2+1] \tag{108}$$

is constant in time, in agreement with general theory.

Let us now examine what happens if we switch on a diffusion $\lambda\partial K_+\partial$ in the expanding direction with strength 2λ. K_+ ist equal to the Jacobi operator, multiplied with the projector on strictly positive eigenvalues.

Explicitly $K_+=diag(\alpha,0)$. The time evolution of the distribution function will now be governed by

$$\frac{d}{dt}\rho-\lambda\alpha\partial_1^2\rho+\alpha(\xi^1\partial_1-\xi^2\partial_2)\rho=0. \tag{109}$$

There exist again Gauss-distributed solutions of the form

$$\rho(\xi,t)=\rho^1(\xi^1,t)\rho_0^2(\xi^2,t) \tag{110}$$

with unchanged second factor, and with

$$\rho^1(\xi^1,t)=(2\pi f^1(t))^{-\frac{1}{2}}\exp[-(\xi^1)^2/2f^1(t)] \quad , \quad f^1(t)=\lambda(e^{2\alpha t}-1)\ . \tag{111}$$

This is the solution which develops from a δ-function at time 0.

An expanding Gauss packet is now found when the starting distribution is a δ-function in ξ^1 at time $t=0$ as well. The width evolves according to

$$\Delta\xi^1=\begin{cases}(Dt)^{\frac{1}{2}} & \text{when } t\mapsto 0\\ \lambda^{\frac{1}{2}}e^{\alpha t} & \text{when } t\mapsto\infty,\end{cases} \tag{112}$$

mit $D=2\alpha\lambda$. We see that the behavior at large t is the same as it was when there was no diffusion but an initial width $\lambda>0$ at time $t=0$.

If we imagine that we may for the purpose of predictions convolute the initial distribution with a Gauss-function of width $\Delta\xi^1=\lambda$ because of finite measuring accuracy, then the introduction of diffusion in the unstable direction has no detectable influence of the dynamics. It only simulates a finite measuring accuracy.

According to Prigogine [2] the thermo-dynamic time evolution should differ from the Hamiltonian one by a term which is odd under time reflection. This is not yet the case here. But it can be achieved by introducing an additional diffusion term also in the stable direction. We shall see that this is permissible in the same spirit of simulating a limited measuring accuracy. We replace K_+ by $K_+ - K_-$ and obtain as an equation for ρ

$$\frac{d}{dt}\rho + \lambda\partial(K_+ - K_-)\partial + \{H,\rho\} = 0 \tag{113}$$

Explicitly,

$$\frac{d}{dt}\rho + \lambda\alpha(\partial_1^2 + \partial_2^2) + \alpha(\xi^1\partial_1 - \xi^2\partial_2)\rho = 0 \ . \tag{114}$$

The solution which evolves from a δ -function at time $t = 0$ is

$$\rho(\xi,t) = \rho^1(\xi^1,t)\rho^2(\xi^2,t) \tag{115}$$

with ρ^1 as before, and

$$\rho^2(\xi^2,t) = (2\pi f^2(t))^{-\frac{1}{2}}\exp[-(\xi^2)^2/2f^2(t)] \quad , \quad f^2(t) = \lambda(1 - e^{-2\alpha t}) \ . \tag{116}$$

The width of the distribution in ξ^2 behaves as follows,

$$\Delta\xi^2 = \begin{cases} (Dt)^{\frac{1}{2}} & \text{when } t \mapsto 0 \\ \lambda & \text{when } t \mapsto \infty, \end{cases} \tag{117}$$

with $D = 2\alpha\lambda$. We see that the width remains now finite and equal to a hypothetical measuring uncertainty λ,

The expression (108) for the entropy is correct for every Gauss distribution. For large t, the entropy now increases linearly with time

$$S(t) = 1 + \frac{1}{2}\ln[(2\pi\lambda)^2(1 - e^{-2\alpha t})(e^{2\alpha t} - 1)] \sim \alpha t \ . \tag{118}$$

Parabolic motion

The free motion of a particle furnishes the prototype of a parabolic dynamics. We write ξ^1 in place of p and ξ^2 in place of q. The Hamiltonian is $H = \frac{1}{2}\beta(\xi^1)^2$. The solution of the equation of motion is

$$\xi^1(t) = \xi^1(0) \quad , \quad \xi^2(t) = \xi^2(0) + \beta t\xi^1(0) \ . \tag{119}$$

The equation of motion for the distribution function has the explicit form

$$\frac{d}{dt}\rho - \beta\xi^1\partial_2\rho = 0 \ . \tag{120}$$

It possesses the Gauss-distributed solution

$$\rho(\xi,t) = 2\pi(\lambda_1\lambda_2)^{-\frac{1}{2}}\exp\{-\frac{1}{2\lambda_1}(\xi^1)^2 + \frac{1}{2\lambda_2}(\xi^2 - \beta t\xi^1)^2\} \ . \tag{121}$$

The probability distribution of the individual coordinates obtains from this as

$$\begin{aligned} p_1(\xi^1) &= \int d\xi^2\rho(\xi^1,\xi^2,t) = (2\pi\lambda_1)^{\frac{1}{2}}\exp\{-\frac{1}{2\lambda_1}(\xi^1)^2\} \ ; && (122)\\ p_2(\xi^2) &= \int d\xi^1\rho(\xi^1,\xi^2,t) = (2\pi f^0(t))^{\frac{1}{2}}\exp\{-\frac{1}{2f^0(t)}(\xi^2)^2\}, && (123)\\ f^0(t) &= \lambda_1(\beta t)^2 + \lambda_2 \ . && (124) \end{aligned}$$

The mean square deviation of ξ^1 remains bounded, while

$$(\Delta\xi^2)^2 = \lambda_1(\beta t)^2 + \lambda_2 \ . \tag{125}$$

An initial uncertainty of ξ^2 alone does not create an uncertainty which grows with time. But an initial uncertainty in momentum ξ^1 creates an uncertainty square in position ξ^2 which grows quadratically with time, but not exponentially as happened in the hyperbolic case.

Let us assume that the measuring uncertainties $\lambda_1 = (\Delta\xi^1)^2$ and $\lambda_2 = (\Delta\xi^2)^2$ at time $t = 0$ obey an uncertainty relation

$$\lambda_1\lambda_2 \geq \frac{\hbar^2}{4} \ . \tag{126}$$

Then it follows from eq. (125), that $(\Delta\xi^1)^2 \geq \frac{\hbar}{2}\beta t$.

Let us tentatively introduce a diffusion term for ξ^1 in the equation of motion for the distribution function.

$$\frac{d}{dt}\rho - \beta\xi^1\partial_2\rho - \frac{D}{2}\partial_1^2\rho = 0 \ . \tag{127}$$

The equation has the Gauss-distributed solution

$$\rho(\xi, t) = \pi(\lambda_1 f(t))^{-\frac{1}{2}} \exp\{-\frac{1}{2\lambda_1}(\xi^1)^2 + \frac{1}{2f(t)}(\xi^2 - \beta t\xi^1)^2\} \ , \tag{128}$$

$$f(t) = Dt + \lambda_1 \ . \tag{129}$$

The distribution of the individual coordinates ξ^i can be computed as before. The uncertainty of ξ^2 is constant in time and for the uncertainty of ξ^1 we find

$$(\Delta\xi^2)^2 = Dt + \lambda_2 + \lambda_1(\beta t)^2 \ . \tag{130}$$

Assuming, the uncertainty relation (126) is valid at time 0, the asymptotic behaviour of $\Delta\xi^2$ for large t remains the same as before.

This could justify the introduction of a diffusion for parabolic motions also under restrictive conditions on the measuring uncertainties. But all this looks very much like a simulation of quantum mechanical effects. Therefore we will not consider it further. We will not introduce diffusion processes for parabolic modes in this paper.

Let us note, however, that parabolic matrices can be obtained from hyperbolic matrices as a limit by a contraction. Therefore parabolic motions can be limits of hyperbolic motions. A situation like this occurs in the Sinai billard. The time evolution operator ϕ^*_{0t} for the deviation is hyperbolic for a short but finite t if scattering occurs at time $0 + \epsilon$. But it becomes parabolic in the limit $t \mapsto 0$. One should think of the scattering process as an idealization of a scattering process which lasts a finite time.

Example: Elliptic motion

The 1-dimensional harmonic oscillator is the prototype of an elliptic motion. The Hamiltonian is

$$H = \frac{1}{2}\beta\{(\xi^1)^2 + (\xi^2)^2\}.$$

A detailed treatment is omitted. It is well known that an initial uncertainty in either ξ^1 or ξ^2 or both will lead to uncertainties which remain bounded for all times.

In contrast to this, the introduction of a diffusion process into equation of motion for the distribution function would lead to an uncertainty which grows with time. Therefore such a diffusion term cannot be justified by the postulate of a finite measuring accuracy.

Conclusion

The introduction of a diffusion process in order to simulate a finite measuring accuracy can be justified in the hyperbolic case. Under restrictive conditions on the measuring uncertainties this is also true for the parabolic case, but not in the elliptic case. It is discussed in the main text how to separate out the hyperbolic modes in case the phase space has more dimensions. In this case the motion is in general not simply either hyperbolic, or parabolic or elliptic, but a combination

A systematic discussion of possible diffusion processes for parabolic modes is not attempted here. It is made difficult also by the nonapplicability of Krein's stability analysis (the symplectic matrix has a zero spectral value). We adopt the attitude that this problem is solved by quantum mechanics.

Appendix D: Poynting vector

The Hamiltonian time development of a virtual composite object, or of any subcategory of K, under the influence of its environment is given by

$$\frac{d}{dt}\xi^\alpha = \{H_O, \xi^\alpha\} + F^\alpha_{ext} \tag{131}$$

$$H_O = \sum_{b\in K_t(O)} H_b \quad , \quad F^\alpha_{ext} = \sum_{b\in\partial K_t(O)} \omega^{\alpha\beta}\frac{\partial H_b}{\partial \xi^\beta} \ . \tag{132}$$

H_O depends only on the state of the objects and arrows in $K_t(O)$ ab. It describes the dynamics in the absence of the influence of the environment. The exterior forces F^α_{ext} describe the influence of the environment. We regard them as given by the state of the system at time t.

The change of energy H_O in O comes out of the equations of motion.

$$\frac{d}{dt}\xi^\alpha = \{H_O, \xi^\alpha\} + F^\alpha_{ext} \tag{133}$$

$$H_O = \sum_{b\in K_t(O)} H_b \quad , \quad F^\alpha_{ext} = \sum_{b\in\partial K_t(O)} \omega^{\alpha\beta}\frac{\partial H_b}{\partial \xi^\beta} \ . \tag{134}$$

It follows that

$$\frac{d}{dt}H_O = -F^\alpha_{ext}\frac{\partial}{\partial\xi^\alpha}H_O \ . \tag{135}$$

According to formula for F^α_{ext} the influx of energy on the right hand side of eq.(135) is a sum of contributions of elementary arrows which point into O They sit on links b. Therefore

$$\frac{d}{dt}H_O = -\sum_{b\in\partial K_t(O)} S_b \tag{136}$$

$$S_b = \frac{\partial H_b}{\partial\xi^\beta}\omega^{\beta\alpha}\frac{\partial}{\partial\xi^\alpha}H_O \tag{137}$$

We call S the *Poynting vector* as in electrodynamics. According to our assumptions about the symplectic matrix only derivatives with respect to such variables occur, which indicate the state of an object at the end of an arrow from outside. They come only from contributions $H_{b'}$ of links b' which point to these objects from inside.

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